# A quick guide on $\infty$-harmonic functions

Kai Xu

There are plenty of references [1, 3, 5, 9, 13] where a comprehensive introduction of $\infty$-harmonic functions can be found (many of them are based primarily on comparison with cones). The purpose of this note is not to provide a general overview in any sense, but to derive the facts in [14, Subsection 2.1] in a quick and self-contained way. All facts stated in this note hold in all dimensions. We begin with the definition:

**Definition 0.1.** *Assume $\Omega$ is a domain in $\mathbb{R}^n$, and $u \in C^0(\Omega)$, and $z \in \Omega$. We say that:*

- *$u$ is $\infty$-subharmonic at $z$, if for all $\varphi \in C^2$ that touches $u$ from above at $z$, we have $\nabla^2\varphi(\nabla\varphi, \nabla\varphi)(z) \geqslant 0$;*
- *$u$ is $\infty$-superharmonic at $z$, if for all $\varphi \in C^2$ that touches $u$ from below at $z$, we have $\nabla^2\varphi(\nabla\varphi, \nabla\varphi)(z) \leqslant 0$;*
- *$u$ is $\infty$-harmonic at $z$, if it is both $\infty$-subharmonic and $\infty$-superharmonic at $z$;*
- *$u$ is $\infty$-harmonic ($\Delta_\infty u = 0$) in $\Omega$, if it is $\infty$-harmonic at every point in $\Omega$.*

We say that $u$ satisfies the *comparison with cone property* if for all $V \Subset \Omega$ and functions of the form $c(x) = a + b|x - x_0|$, we have

$$u \leqslant c \text{ (resp. } u \geqslant c) \text{ on } \partial(V \setminus \{x_0\}) \qquad \Rightarrow \qquad u \leqslant c \text{ (resp. } u \geqslant c) \text{ in } V.$$

The basic theory [6, 5] and regularity theorems of Evans-Savin and Savin in dimension 2 [7, 11] depend much on this property.

**Lemma 0.2.** *$\infty$-harmonicity $\Rightarrow$ comparison with cones.*

*Proof.* Suppose $c(x) = a + b|x - x_0|$, and $V \Subset \Omega$, and $u \leqslant c$ on $\partial(V \setminus \{x_0\})$ but $u \nleqslant c$ in $V$. This implies

$$\min_{V\setminus\{x_0\}}(c - u) < \min_{\partial(V\setminus\{x_0\})}(c - u).$$

If $b > 0$, then for a $s < 1$ close enough to 1, there is a constant $\delta$ so that $c'(x) = a + b|x - x_0|^s + \delta$ touches $u$ from above at a point in $V \setminus \{x_0\}$. However, we may calculate

$$\nabla^2 c'(\nabla c', \nabla c') = b^3 s^3 (s - 1)|x - x_0|^{3s-4} < 0,$$

thus it contradicts $\infty$-subharmonicity. If $b < 0$, then consider the same function but with $s > 1$. If $b = 0$, then there are constants $0 < \varepsilon_1 \ll \varepsilon_2 \ll 1$ and $\delta > 0$ and a unit vector field $e$, so that the function $c(x) = a + \varepsilon_1\langle x\,,\, e\rangle - \varepsilon_2|x|^2/2$ touches $u$ from above at a point $x_0 \in V$, and

$$\nabla^2 c(\nabla c, \nabla c) = -\varepsilon_2|\varepsilon_1 e - \varepsilon_2 x|^2 < 0 \qquad \text{in } \overline{V}.$$

This contradicts the $\infty$-subharmonicity of $u$ as well. $\square$

**Lemma 0.3.** *If $\Delta_\infty u = 0$ in $\Omega \subset \mathbb{R}^n$, then*

$$\max_K(u) = \max_{\partial K}(u), \qquad \min_K(u) = \min_{\partial K}(u), \qquad \forall\, K \Subset \Omega.$$

*Proof.* See the $b = 0$ case of the proof of Lemma 0.2. $\square$

**Lemma 0.4.** *If $\Delta_\infty u = 0$ in $\Omega \subset \mathbb{R}^n$, then $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, and*

$$|\nabla u(x_0)| \leqslant \frac{2\sup_\Omega |u|}{d(x_0, \partial\Omega)}, \qquad \forall\, a.e.\ x_0 \in \Omega.$$

*Proof.* For each $r < d(x_0, \partial\Omega)$, we may compare $u$ with $c(x) = u(x_0) \pm 2r^{-1}|x - x_0| \sup_\Omega |u|$ in the region $V = B(x_0, r) \Subset \Omega$, and obtain $|\nabla u(x_0)| \leqslant 2r^{-1} \sup_\Omega |u|$. $\square$

For a point $x_0 \in \Omega$, denote

$$\mathrm{Lip}_u(x_0) = \limsup_{x \to x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|},$$

and for each $0 < r < d(x_0, \partial\Omega)$, denote

$$S_u^+(x_0, r) = \max_{|x - x_0| = r} r^{-1}\big[u(x) - u(x_0)\big], \qquad S_u^-(x_0, r) = \max_{|x - x_0| = r} r^{-1}\big[u(x_0) - u(x)\big].$$

The following facts are fundamental:

$$\left.\begin{array}{l} 0 \leqslant S_u^+(x_0, r) \leqslant S_u^+(x_0, R) \\ 0 \leqslant S_u^-(x_0, r) \leqslant S_u^-(x_0, R) \end{array}\right\} \qquad \forall\, r < R < d(x_0, \partial\Omega), \tag{0.1}$$

and

$$\mathrm{Lip}_u(x_0) = \lim_{r \to 0} S_u^+(x_0, r) = \lim_{r \to 0} S_u^-(x_0, r). \tag{0.2}$$

The proofs can also be found in [6, Lemma 2.4−2.7] or [5, Lemma 4.1−4.3].

*Proof of* (0.1).

To prove $S_u^+(x_0, r) \leqslant S_u^+(x_0, R)$, one compares $u$ with the cone function

$$c(x) = u(x_0) + S_u^+(x_0, R)|x - x_0|$$

in $V = B(x_0, R)$. The fact $S_u^+(x_0, r) \geqslant 0$ follows from Lemma 0.3. $\square$

*Proof of* (0.2).

WLOG $u(x_0) = 0$. Denote the two limits in (0.2) by $S^\pm$: they exist and are nonnegative by (0.1). Clearly

$$\mathrm{Lip}_u(x_0) = \max\{S^+, S^-\}.$$

It remains to show $S^+ = S^-$. If $S^+ > S^-$, then $S^+ > S_u^-(x_0, r) + \delta$ for some $r, \delta \ll 1$. For $\rho < \min\{r/2, r\delta/2S^+\}$, we find a point $x_\rho \in \partial B(x_0, \rho)$ that maximizes $u$ in $\overline{B}(x_0, \rho)$. Then

$$u(x_\rho) = S_u^+(x_0, \rho)\rho \geqslant S^+\rho.$$

Consider the cone function

$$c(x) = u(x_\rho) - \frac{u(x_\rho) + rS_u^-(x_0, r)}{r - \rho}|x - x_\rho|.$$

Then $c(x_\rho) = u(x_\rho)$, and $c \leqslant u$ on $\partial B(x_0, r)$. Then comparison with cone implies

$$\begin{aligned} 0 = u(x_0) \geqslant c(x_0) &= u(x_\rho) - \frac{u(x_\rho) + rS_u^-(x_0, r)}{r - \rho}\rho = \frac{r - 2\rho}{r - \rho}u(x_\rho) - \frac{r\rho}{r - \rho}S_u^-(x_0, r) \\ &\geqslant \frac{r - 2\rho}{r - \rho}S^+\rho - \frac{r\rho}{r - \rho}(S^+ - \delta) = \frac{\rho}{r - \rho}(-2\rho S^+ + r\delta) > 0, \end{aligned}$$

contradiction. Thus $S^+ \leqslant S^-$. Switching the sign of $u$, this shows $S^- \leqslant S^+$ as well. $\square$

The $\infty$-Laplacian satisfies the weak maximum principle:

**Lemma 0.5.** *If $u, v \in C^0(\overline{\Omega})$ are respectively $\infty$-subharmonic and $\infty$-superharmonic in $\Omega$, then*

$$\max_{\overline{\Omega}}(u-v) = \max_{\partial\Omega}(u-v).$$

*Proof.* This was first proved by Jensen [8]. Later, Armstrong-Smart [2] discovered a short proof that relies only on comparison with cones. $\square$

Then, $\infty$-harmonic functions are the $C^0_{\mathrm{loc}}$ limit of $p$-harmonic functions as $p \to \infty$:

**Lemma 0.6.** *Suppose $\Omega \subset \mathbb{R}^n$ and $u \in C^0(\Omega)$. Then $u$ is $\infty$-harmonic in $\Omega$ if and only if $u$ is the $C^0_{\mathrm{loc}}(\Omega)$ limit of a sequence of $p_i$-harmonic functions, with $p_i \to \infty$.*

*Proof.* (cf. [4, 8, 10].) Suppose $u_i$ are $p_i$-harmonic, with $p_i \to \infty$ and $u_i \to u$ in $C^0_{\mathrm{loc}}(\Omega)$. If $u$ is not $\infty$-subharmonic, then there is a point $x_0 \in \Omega$ and a $C^2$ test function $\varphi$ in a neighborhood $V \ni x_0$, so that

$$\varphi(x_0) = u(x_0), \qquad \varphi > u \ \text{ in } \ \overline{V} \setminus \{x_0\}, \qquad \nabla^2\varphi(\nabla\varphi, \nabla\varphi)(x_0) < 0. \tag{0.3}$$

Hence $\min_{\overline{V}}(\varphi - u_i) < \min_{\partial V}(\varphi - u_i)$ for all large enough $i$. So there are points $x_i \in V$ which minimize $\varphi - u_i$ in $\overline{V}$. By (0.3) we have $x_i \to x_0$. The $p_i$-harmonicity of $u_i$ implies

$$0 \leqslant \Delta_p\varphi(x_i) = \Big[|\nabla\varphi|^{p_i-2}\Delta\varphi + (p_i-2)|\nabla\varphi|^{p_i-4}\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\Big]\Big|_{x=x_i}.$$

Taking $i \to \infty$, we obtain $\nabla^2\varphi(\nabla\varphi,\nabla\varphi)(x_0) \geqslant 0$, contradiction.

On the other hand, fix an $\infty$-harmonic function $u$ in $\Omega$. Recall $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Take a smooth exhaustion $K_1 \Subset K_2 \Subset \cdots \Subset \Omega$ of $\Omega$. For each $K_i$, consider the Dirichlet problem

$$\begin{cases} \Delta_p u_{i,p} = 0 & \text{in } \ K_i, \\ u_{i,p} = u & \text{on } \ \partial K_i. \end{cases}$$

The minimization of Dirichlet energy gives

$$\|\nabla u_{i,p}\|_{L^p(K_i)} \leqslant \|\nabla u\|_{L^p(K_i)} \leqslant |K_i|^{1/p}\|\nabla u\|_{L^\infty(K_i)}.$$

Hence, there is a sequence $p \to \infty$ along which $u_{i,p} \to u'$ in $W^{1,q}(K_i) \cap C^{0,\alpha}(K_i)$ ($\forall\, q > 1$ and $\alpha < 1$), for some function $u'$ that is $\infty$-harmonic in $K_i$. A boundary barrier argument implies that $u' = u$ on $\partial K$. Hence $u' = u$ by Lemma 0.5. As the limit is now unique, we have $u_{i,p} \to u$ in $C^0(K_i)$. Running this argument for each $i$ and choosing a diagonal sequence, we may find $p_i \to \infty$ so that $\|u_{i,p_i} - u\|_{C^0(K_i)} \leqslant i^{-1}$, $\forall\, i$, as desired. $\square$

**Lemma 0.7.** *If $L > 0$, and $u$ is $\infty$-harmonic and $L$-Lipschitz in $\Omega$, then the set*

$$\Big\{x_0 \in \Omega : \mathrm{Lip}_u(x_0) = L\Big\}$$

*is a disjoint union of nontrivial line segments which are streamlines of $u$. Each segment does not have endpoints in $\Omega$.*

*Proof.* (cf. [6, 5, 12].) Suppose $\mathrm{Lip}_u(x_0) = L$. For any $r$ with $B(x_0, r) \Subset \Omega$, consider $x_1, x_2$ that maximizes and minimizes $u$ in $\overline{B}(x_0, r)$, respectively. Then

$$Lr = L|x_i - x_0| \geqslant |u(x_i) - u(x_0)| \geqslant r S_u^{\pm}(x_0, r) \geqslant r\,\mathrm{Lip}_u(x_0) = Lr, \qquad i = 1, 2,$$

hence $u(x_i) - u(x_0) = \pm Lr$. Denote $e = \frac{x_1 - x_0}{|x_1 - x_0|}$, hence $x_1 = x_0 + re$. Then using the $L$-Lipschitzness of $u$, it is elementary to verify that $x_2 = x_0 - re$ and

$$u(x_0 + ce) = u(x_0) + cL, \quad \nabla u(x_0 + ce) = Le, \qquad \forall\, c \in (-r, r),$$

hence $\{x_0 + ce\}_{c \in [-r,r]}$ is a streamline of $u$ that passes through $x_0$.

We have shown that $u$ is differentiable at each $x_0 \in \{\mathrm{Lip}_u = L\}$, and for each disk $B(x_0, r) \Subset \Omega$ the segment $\{x_0 + cL^{-1}\nabla u(x_0)\}_{c \in (-r,r)}$ is a streamline of $u$. Then, clearly, a maximal segment has no endpoints in $\Omega$, and different maximal segments are disjoint. ☐

**Lemma 0.8.** *If $\Delta_\infty u = 0$ in $\Omega \subset \mathbb{R}^n$, and $K \Subset \Omega$, then $\sup_K |\nabla u| = \sup_{\partial K} |\nabla u|$.*

*Proof.* Lemma 0.7 implies $\big\{x \in K : |\nabla u(x)| = \sup_K |\nabla u|\big\} \not\Subset K$ if it is nonempty. ☐

Then we prove the removable singularity theorem:

**Lemma 0.9.** *If $u \in C^1(\Omega)$ is $\infty$-harmonic in $\Omega \setminus \{x_0\}$, then $u$ is $\infty$-harmonic in $\Omega$.*

*Proof.* (cf. [12, Lemma 4.2].) Suppose $u$ is not $\infty$-subharmonic at $x_0$ (same argument for superharmonicity). Then there a $C^2$ test function $\varphi$ in some neighborhood $V \ni x_0$, so that

$$\varphi(x_0) = u(x_0), \qquad \varphi > u \ \text{ in } \ V \setminus \{x_0\}, \qquad \nabla^2\varphi(\nabla\varphi, \nabla\varphi)(x_0) < 0. \tag{0.4}$$

Then clearly $\nabla\varphi(x_0) = \nabla u(x_0)$. For any unit vector $e$ and small $\varepsilon > 0$, there is a constant $c_\varepsilon$ so that

$$\varphi_\varepsilon(x) = \varphi(x) - \varepsilon\langle x\,,\, e\rangle + c_\varepsilon$$

touches $u$ from above at some point $x_\varepsilon \in V$. Then $x_\varepsilon \neq x_0$ since $\nabla\varphi_\varepsilon(x_0) \neq \nabla u(x_0)$, and $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$ due to (0.4). The $\infty$-subharmonicity of $u$ in $\Omega \setminus \{x_0\}$ implies

$$\nabla^2\varphi_\varepsilon(\nabla\varphi_\varepsilon, \nabla\varphi_\varepsilon)(x_\varepsilon) \geqslant 0.$$

But as $\varepsilon \to 0$ this implies $\nabla^2\varphi(\nabla\varphi, \nabla\varphi)(x_0) \geqslant 0$, contradicting (0.4). ☐

# References

[1] D. Araujo, J. Urbano, *The $\infty$-Laplacian: from AMLE to machine learning*, No. 34 Colóquio Brasileiro de Matemática, IMPA, 2023.

[2] S. Armstrong, C. Smart, *An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions*, Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 381–384.

[3] G. Aronsson, M. Crandall, P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505.

[4] T. Bhattacharya, E. DiBeneditto, J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u = f$ and related extremal problems*, Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale 1989 Nonlinear PDE's, 15-68.

[5] M. G. Crandall, *A visit with the $\infty$-Laplace equation*, Calculus of Variations and Non-Linear Partial Differential Equations, Lecture Notes in Mathematics 1927, Springer, Berlin, Heidelberg, pp. 75–122.

[6] M. G. Crandall, L. C. Evans, R. F. Gariepy, *Optimal Lipschitz extensions and the infinity laplacian*, Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139.

[7] L. C. Evans, O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations 32 (2008), no. 3, 325–347.

[8] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal. 123 (1993), no. 1, 51-74.

[9] P. Lindqvist, *Notes on the Infinity Laplace equation*, SpringerBriefs Math, BCAM Basque Center for Applied Mathematics, Bilbao; Springer, [Cham], 2016. ix+68 pp.

[10] P. Lindqvist, J. Manfredi, *The Harnack inequality for $\infty$-harmonic functions*, Electron. J. Differential Equations 1995, No. 04, approx. 5 pp.

[11] O. Savin, *$C^1$ regularity for infinity harmonic functions in two dimensions*, Arch. Ration. Mech. Anal. 176 (2005), no. 3, 351–361.

[12] O. Savin, C. Wang, Y. Yu, *Asymptotic behavior of infinity harmonic functions near an isolated singularity*, Int. Math. Res. Not. IMRN 2008 (6), Art. ID rnm163, 23 pp.

[13] C. Wang, *An Introduction of Infinity Harmonic Functions*, unpublished notes, 2008.

[14] K. Xu, *Infinity-harmonic functions and inverse mean curvature flow clusters*, preprint (2026), arxiv:2607.06698.