Harmonic functions and monotonicity formulas

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Consider a harmonic function u on a 3-manifold M. It is now known that studying the evolution of certain quantities along the level sets of u has far-reaching geometric and topological consequences. This note intends to provide a streamlined introduction to this topic that summarizes various past results.

Denote

$$\Sigma_t = \{u = t\}, \qquad J(t) = \int_{\Sigma_t} |\nabla u|^2.$$

One of the central formulas is the second order ODE inequality

$$J''(t) \geqslant \frac{3}{4}\frac{J'(t)^2}{J(t)} - 2\pi\chi(\Sigma_t), \tag{0.1}$$

assuming nonnegative scalar curvature of the ambient space (see Lemma 0.4 for the precise form). The power 2 in $J(t)$ is chosen so that the Gauss-Bonnet formula applies.

Example 0.1. It is useful to calculate the model case $u = r^{-1} + \beta$ in $\mathbb{R}^3$: for each t we have

$$\Sigma_t = \Big\{r = \frac{1}{t-\beta}\Big\}, \qquad |\nabla u| = \frac{1}{r^2} = (t-\beta)^2 \ \text{ on } \ \Sigma_t,$$

hence

$$J(t) = 4\pi(t-\beta)^2 \qquad \forall\, t \in (\beta, \infty).$$

This is a case of equality for (0.1).

Assuming additionally that u is a Green's kernel and Σ_t are all connected, (0.1) may be integrated from t to ∞ to obtain a first order monotonicity

$$tJ'(t) \leqslant J(t) + 4\pi t^2 \qquad \text{or} \qquad \frac{d}{dt}\Big(\frac{J(t)}{t}\Big) \leqslant 4\pi. \tag{0.2}$$

This recovers the result of Munteanu-Wang [4, Theorem 1.1], and in the proof, we also recover a theorem by Agostiniani-Mazzieri-Oronzio [1, Theorem 1.1].

If u is the Dirichlet Green's kernel in an ambient domain, or if u is a minimal positive Green's kernel on a strongly nonparabolic manifold, then we may bound $J(0^+)$ from below using (0.1) (0.2) and other known conditions. By doing this, we recover the integral gradient estimates by Colding-Minicozzi [3, Theorem 0.3] and Yan-Zhu [6, Lemma 3.10], and also recover the boundary energy estimate by You-Zhang [7, Proposition 4.8, 4.9]. The last result is crucial in their recent proof of Yau and Gromov's conjecture (that a complete contractible PSC 3-manifold is diffeomorphic to $\mathbb{R}^3$). One may also choose to derive monotonicity formulas on the function $b = 1/u$ instead of u, and the resulting differential inequality is equivalent to (0.1) up to change of variables and algebraic deformations.

We begin with a most general setting, assuming an connected oriented ambient 3-manifold with nonnegative scalar curvature, and a harmonic function u and a time interval (a,b), so that $u : u^{-1}((a,b)) \to (a,b)$ is proper. For each $t \in (a,b)$, define

$$J(t) = \int_{\Sigma_t \setminus \mathrm{Crit}(u)} |\nabla u|^2.$$

For all regular values $t \in (a,b)$, define

$$K(t) = \int_{\Sigma_t} \nabla^2 u(\nu,\nu).$$

We need an analytic lemma:

Lemma 0.2 ([2, Lemma 2.4])**.** *$J \in \mathrm{Lip}_{\mathrm{loc}}((a,b))$.*

At each point where $\nabla u \neq 0$, we denote $\nu = \nabla u/|\nabla u|$ and fix an orthonormal frame $\{e_1, e_2, e_3\}$ with $e_1 = \nu$. Assume the indices $i, j \in \{2,3\}$. Recall the standard formula for the second fundamental form and mean curvature of Σ_t:

$$\begin{aligned} A_{ij} &= \nabla_i\Big(\frac{\nabla_j u}{|\nabla u|}\Big) = \frac{\nabla_{ij} u}{|\nabla u|}, \\ H &= \mathrm{div}\,\Big(\frac{\nabla u}{|\nabla u|}\Big) = \frac{\Delta u - \nabla^2 u(\nu,\nu)}{|\nabla u|} = -\frac{\nabla_{11} u}{|\nabla u|}. \end{aligned} \tag{0.3}$$

Lemma 0.3. *$J'(t) = K(t)$ for all regular t.*

Proof. As the family $\{\Sigma_t\}$ evolves smoothly with the normal speed $1/|\nabla u|$ near any regular value, the usual smooth calculation and $\Delta u = 0$ give

$$J'(t) = \int_{\Sigma_t} \Big[\big\langle \nabla |\nabla u|^2, \nu\big\rangle + H|\nabla u|^2\Big] \frac{1}{|\nabla u|} = K(t). \qquad \square$$

The main monotonicity formula is as follows:

Lemma 0.4. *For all regular $s < t \in (a,b)$ it holds*

$$K(t) - K(s) \geqslant \int_s^t \Big[\frac{3}{4}\frac{K(\tau)^2}{J(\tau)} - 2\pi\chi(\Sigma_\tau)\Big]\, d\tau. \tag{0.4}$$

Proof. Notice that

$$K(t) = \int_{\Sigma_t} \big\langle \nabla|\nabla u|, \nu\big\rangle.$$

Denoting $\Omega = \{s < u < t\}$, we have

$$K(t) - K(s) = \int_{\partial\Omega} \langle \nabla|\nabla u|, \nu_\Omega\rangle = \int_\Omega \Delta|\nabla u|.$$

For a general smooth vector field X, the distributional Laplacian $\Delta|X|$ is a signed measure, and can be decomposed into $\big(\Delta|X| \cdot \mathcal{L}^n\big) \llcorner \{X \neq 0\}$ and a possibly singular nonnegative measure. (It is shown in [7, Lemma 5.4] that $|\nabla u| \in W^{2,1}_{\mathrm{loc}}$ since u is harmonic, but we do not need this fact here.)

In the set $\{\nabla u \neq 0\}$, Bochner's formula gives

$$\Delta|\nabla u| = \frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|} + \mathrm{Ric}(\nu,\nu)|\nabla u|. \tag{0.5}$$

The Gauss formula gives

$$\mathrm{Ric}(\nu,\nu) = \Big(\frac{R}{2} - K_{\Sigma_t}\Big) + \frac{1}{2}\big(H^2 - |A|^2\big), \tag{0.6}$$

where R is the scalar curvature of M and K_{Σ_t} is the Gauss curvature of Σ_t. Using (0.3) and writing $u_{ij} = \nabla^2 u(e_i, e_j)$, we obtain

$$H^2 - |A|^2 = \frac{1}{|\nabla u|^2}\big[u_{11}^2 - u_{ij}^2\big] \tag{0.7}$$

and

$$\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|} = \frac{1}{|\nabla u|}\Big[\big(u_{11}^2 + u_{ij}^2 + 2u_{1i}^2\big) - \big(u_{11}^2 + u_{1i}^2\big)\Big]. \tag{0.8}$$

Finally, notice that

$$u_{ij}^2 \geqslant \frac{1}{2}(u_{ii})^2 = \frac{1}{2}u_{11}^2. \tag{0.9}$$

Combining (0.5) $\sim$ (0.9) yields on $\{\nabla u \neq 0\}$

$$\Delta|\nabla u| \geqslant \frac{1}{2}\frac{u_{ij}^2}{|\nabla u|} + \frac{1}{2}\frac{u_{11}^2}{|\nabla u|} + \Big(\frac{R}{2} - K_{\Sigma_t}\Big)|\nabla u| \geqslant \frac{3}{4}\frac{u_{11}^2}{|\nabla u|} + \Big(\frac{R}{2} - K_{\Sigma_t}\Big)|\nabla u|. \tag{0.10}$$

(A refined Kato's inequality is implicitly contained here.) Hence

$$\begin{aligned} K(t) - K(s) &\geqslant \int_\Omega \frac{3}{4}\frac{u_{11}^2}{|\nabla u|} + \Big(\frac{R}{2} - K_{\Sigma_\tau}\Big)|\nabla u| \\ &= \int_s^t \int_{\Sigma_\tau} \Big[\frac{3}{4}\frac{u_{11}^2}{|\nabla u|^2} + \frac{R}{2} - K_{\Sigma_\tau}\Big]\, d\tau \geqslant \int_s^t \Big[\frac{3}{4}\frac{K(\tau)^2}{J(\tau)} - 2\pi\chi(\Sigma_\tau)\Big]\, d\tau. \quad \square \end{aligned}$$

Lemma 0.5. *If Σ_t are connected for all regular $t \in (a,b)$, then J can be written as the sum of a convex function and a $W^{2,\infty}_{\mathrm{loc}}$ function in (a,b).*

Proof. Set

$$g(t) = \frac{3}{4}\frac{J'(t)^2}{J(t)} - 4\pi \in L^\infty_{\mathrm{loc}}((a,b)).$$

For a fixed $t_0 \in (a,b)$ and for all regular values t, set

$$J_1(t) = J'(t) - \int_{t_0}^t g(\tau)\, d\tau \in L^\infty_{\mathrm{loc}}((a,b)).$$

By Lemma 0.4, J_1 is nondecreasing in the set of regular values. Now,

$$J(t) = J(t_0) + \int_{t_0}^t J_1(s)\, ds + \int_{t_0}^t \int_{t_0}^s g(\tau)\, d\tau\, ds \qquad \forall\, t \in (a,b).$$

The second term is convex, and the third term is $W^{2,\infty}_{\mathrm{loc}}$. $\square$

As a result, the distributional derivative of J' is a measure, with

$$dJ' \geqslant \Big[\frac{3}{4}\frac{(J')^2}{J} - 2\pi\chi\Big] \cdot \mathcal{L}^1, \tag{0.11}$$

whenever Σ_t are all connected. An analogous result can also be derived if Σ_t have a bounded number of connected components.

Now, for a basepoint p, we consider G a Green's kernel with pole p, normalized so that $\Delta G = -4\pi\delta_p$. The asymptotic expansion at the pole gives

$$G = r^{-1} + \beta + O(r), \qquad \nabla G = \nabla r^{-1} + O(1), \qquad \nabla^2 G = \nabla^2 r^{-1} + O(r^{-1}), \tag{0.12}$$

where $r = d(\cdot, p)$.

The following recovers Munteanu-Wang's result [4, Theorem 1.1]. Here we do not require strong nonparabolicity of M, nor involve behaviors of G at infinity. The monotonicity of the function $t \mapsto t^{-1}(S - 4\pi)$ in the proof recovers the result of Agostiniani-Mazzieri-Oronzio [1, Theorem 1.1], through a change of variable. Note that $J(t)$ is left and right differentiable by Lemma 0.5. The proof of Lemma 0.6 in fact shows that

$$\frac{d^{\pm}}{dt}\Big[t^{-1}J(t)\Big] \leqslant 4\pi \qquad \forall\, t > a.$$

Lemma 0.6. *Suppose $a > 0$, and G is a Green's kernel with pole p, so that $\{G > a\} \Subset M$ and Σ_t are connected for all regular $t > a$. Then*

$$\frac{d}{dt}\Big[t^{-1}J(t)\Big] \leqslant 4\pi \qquad \forall \text{ regular } t > a.$$

Proof. The quantity

$$S = \frac{d}{dt}(t^{-1}J) = t^{-1}J' - t^{-2}J \tag{0.13}$$

is a $\mathrm{BV}_{\mathrm{loc}}$ function, by Lemma 0.5. The standard BV calculus and (0.11) give

$$\begin{aligned} d(t^{-1}S) = d\big[t^{-2}J' - t^{-3}J\big] &= t^{-2}dJ' + \Big[-2t^{-3}J' + 3t^{-4}J - t^{-3}J'\Big] \\ &\geqslant t^{-2}\Big[\frac{3}{4}\frac{(J')^2}{J} - 2\pi\chi\Big] + \Big[-2t^{-3}J' + 3t^{-4}J - t^{-3}J'\Big] \\ &= \frac{3}{t^2 J}\Big[\frac{J'}{2} - \frac{J}{t}\Big]^2 - \frac{2\pi\chi}{t^2}. \end{aligned}$$

Using (0.12), we may calculate

$$J(t) = 4\pi t^2 + o(t^2), \qquad K(t) = 8\pi t + o(t) \qquad \text{as } \ t \to \infty,$$

hence

$$\lim_{t\to\infty} t^{-1}S(t) = 0. \tag{0.14}$$

Using the connectedness of Σ_t, we have

$$S(t) = -t\int_t^\infty d(t^{-1}S) \leqslant t\int_t^\infty \frac{2\pi\chi(\Sigma_s)}{s^2}\,ds \leqslant t\int_t^\infty \frac{4\pi}{s^2}\,ds = 4\pi. \qquad \square$$

The following lemma essentially recovers Proposition 4.8, 4.9 in You-Zhang [7]. The proof is (nonobviously) equivalent to the one in [7] as well.

Lemma 0.7. *Suppose $T > 0$ and $\Omega \Subset M$ is a smooth bounded domain, with $R \geqslant 0$ in Ω. Let G be the Green's kernel with a pole $p \in \Omega$ and with Dirichlet condition on $\partial\Omega$. Suppose that Σ_t are connected for all $t \in [0,\infty)$, and $\chi(\Sigma_t) \leqslant -2$ for all $t \in [0,T]$. Then*

$$J(0) \geqslant \frac{81\pi}{2^{12}} T^2. \tag{0.15}$$

Proof. Using $\chi(\Sigma_t) \leqslant -2$ for $t \leqslant T$ and $\chi(\Sigma_t) \leqslant 2$ for $t \geqslant T$, the end of Lemma 0.6 gives

$$S(t) \leqslant t\Big[-\int_t^T \frac{4\pi}{s^2}\, ds + \int_T^\infty \frac{4\pi}{s^2}\, ds\Big] = \frac{8\pi t}{T} - 4\pi, \qquad \forall \,\text{regular } t \in (0,T). \tag{0.16}$$

Integrating this in $[T/4, T/2]$ and using $J \geqslant 0$, we obtain

$$\frac{J(T/4)}{T/4} = \frac{J(T/2)}{T/2} - \int_{T/4}^{T/2} S(t)\, dt \qquad \Rightarrow \qquad J(T/4) \geqslant \frac{\pi}{16} T^2. \tag{0.17}$$

Taking $t \searrow T/4$ of (0.16) yields

$$\frac{d^+}{dt}(t^{-1}J)\Big|_{t=T/4} < 0 \qquad \Rightarrow \qquad J'^+(T/4) \leqslant \frac{J(T/4)}{T/4}. \tag{0.18}$$

Finally, in $(0,T)$ we have $dJ' \geqslant \frac{3}{4}\frac{(J')^2}{J}$, which implies

$$d\big[(J^{1/4})'\big] = d\Big[\frac{1}{4} J^{-3/4} J'\Big] = \frac{1}{4} J^{-3/4}\Big[-\frac{3}{4} J^{-1}(J')^2 + dJ'\Big] \geqslant 0.$$

Hence $J^{1/4}$ is convex in $(0,T)$. Combining the convexity and (0.18) we have

$$J(0)^{1/4} \geqslant J(T/4)^{1/4} + \Big[\frac{1}{4} J(T/4)^{-3/4} J'^+(T/4)\Big] \cdot (-T/4) \geqslant \frac{3}{4} J(T/4)^{1/4}.$$

Combining this with (0.17), we get (0.15) as desired. □

The following recovers Colding-Minicozzi's integral gradient estimate [3, Theorem 0.3].

Lemma 0.8. *Suppose $b_2(M) = 0$, and G is a positive Green's kernel on M satisfying*

$$\lim_{x\to\infty} G(x) = \lim_{x\to\infty} |\nabla G(x)| = 0.$$

Then $J(t) \leqslant 4\pi t^2$ for all $t \in (0,\infty)$.

Proof. The topological condition implies that Σ_t is connected for all $t \in (0,\infty)$. The gradient decay implies

$$J(t) = \int_{\Sigma_t} |\nabla G|^2 \leqslant \sup_{\Sigma_t} |\nabla G| \cdot \int_{\Sigma_t} |\nabla G| = o(1)\cdot 4\pi \qquad \text{as} \quad t \to 0.$$

The asymptotics at the pole give

$$\lim_{t\to\infty} t^{-2} J(t) = 4\pi.$$

If $J(t) > 4\pi t^2$ for some $t \in (0,\infty)$, then there is $\lambda > 4\pi$ and $\varepsilon > 0$ so that $J(t)$ touches $f(t) = \lambda(t+\varepsilon)^2$ from below at an interior point $t_0 \in (0,\infty)$. But we may calculate

$$f''(t) = 2\lambda < 3\lambda - 4\pi = \frac{3}{4}\frac{f'(t)^2}{f(t)} - 4\pi, \qquad \forall\, t \in (0,\infty).$$

This contradicts (0.1) and the viscosity principle. □

References

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